

Energy transport in strongly disordered superconductors and magnets

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We develop an analytical theory for quantum phase transitions driven by disorder in magnets and superconductors. We study these transitions with a cavity approximation which becomes exact on a Bethe lattice with large branching number. We find two different disordered phases, characterized by very different relaxation rates, which both exhibit strong inhomogeneities typical of glassy physics.

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The zero temperature quantum phase transitions and their quantum critical points have been rather well understood in translationally invariant systems[1]. Much less is known about disordered systems where the transition is driven by the competition between strong disorder and interactions.

Motivated by experiments on disordered superconductors [2], we formulate a theoretical model of disorder driven transitions and solve it in the simplest controlled approximation. Our main results can also be relevant for other disordered quantum problems, especially for strongly disordered magnets. The new physics introduced by the strong disorder is the appearance of new phases [3] in which all or some excitations are localized in space and have infinite lifetime and thus cannot contribute to any transport. The quantum critical point at which the long range order appears has many features that distinguish it from conventional quantum critical points in translationally invariant systems. Most notably, it is characterized by a wide distribution of the order parameter and the appearance of a new intermediate phase in which only low energy local excitations have infinitely long lifetime while high energy ones can decay.

Strongly disordered films of InO, TiN or Be display a transition from the superconductor to insulator when their resistivity in the normal state exceeds a value of the order of resistance quantum $R_Q = 6.5k\Omega$ [4]. Close to the transition the superconductor-insulator phase can also be induced by magnetic field; this transition displays a quantum critical point behavior [5]. In the vicinity of the quantum critical point, the tunneling spectroscopy shows a well defined gap at all points. However, the coherence peaks expected for a BCS superconductor appear at some locations and disappear at others, a phenomenon which is similar to some experiments in high T_C oxides. The absence of a coherence peak combined with the detection of a superconducting gap in a single electron tunneling exper-

iment implies that the disorder does not affect the local Cooper pairing of electrons but prevents the formation of a coherent state of these pairs. This allows to exclude ‘fermionic’ mechanisms of the superconductivity suppression through a reduction of phonon attraction by Coulomb interactions. Because Coulomb interaction is strongly suppressed in the insulating phase[6], the most plausible mechanism for the superconductor-insulator transition in homogeneous disordered films is a competition between pair hopping and random pair energies on different sites, as suggested in a seminal paper of Ma and Lee[7].

As we show in this paper the solution of this model, which requires going beyond the simple mean-field analysis of the earlier works, reproduces correctly the most important features of the data on disordered films: direct superconductor-insulator transition, activated behavior close to the quantum critical point in the insulating phase, strong dependence of the activation energy near the quantum critical point and huge order parameter variations from site to site in the superconducting phase.

In the absence of Coulomb repulsion the electrons are paired even on localized single electron states, and pairs can hop from one site to another. This physics is described by a Hamiltonian of disordered bosons with strong on-site repulsion [7, 8, 9]:

$$H = - \left(\sum_i \xi_i \sigma_i^z + \sum_{(ij)} M_{ij} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+) \right) \quad (1)$$

Here the state with $\sigma_i^z = \pm 1$ corresponds to a local level occupied or unoccupied by a Cooper pair; ξ_j ’s are occupation energies for each site, which are quenched random variables drawn from a probability $P(\xi)$. M_{ij} describe the pair hopping amplitude between sites i and j . These hopping amplitudes couple a typical local level to a large number of neighbors, $Z \gg 1$. We shall assume that each site is coupled to Z neighbours with $M_{ij} = g/(Z-1)$, and

for technical simplicity we shall study the slightly simplified Hamiltonian

$$H = - \left(\sum_i \xi_i \sigma_i^z + \frac{g}{Z-1} \sum_{(ij)} \sigma_i^x \sigma_j^x \right) \quad (2)$$

but all our conclusions also hold for the case (1). With a redefinition of the meaning of occupied and empty, one can take the site energies ξ_j to be all positive. We shall assume that the ξ are uniformly distributed on the interval $[0, 1/\nu]$. The important feature of this distribution is that it is constant near to $\xi = 0$; the value of ν just sets the scale of energies, and we can choose $\nu = 1$. In this language of Hamiltonians (1,2) the superconducting phase is mapped onto the phase with spontaneous magnetization in the x direction, in the insulating phase spins point parallel to z -axis.

The most obvious approach to study this Hamiltonian is through a simple mean-field (SMF) approach, where H is replaced by $H_{MF} = \sum_i (-\xi_i \sigma_i^z - B \sigma_i^x)$ and B is determined self-consistently. At temperature $T = 1/\beta$, this predicts a phase transition from insulator to superconductor at the critical value of the hopping

$$g_c^{SMF} = \left(\int d\xi P(\xi) \tanh(\beta\xi)/\xi \right)^{-1}. \quad (3)$$

As $P(0) > 0$, $g_c^{SMF} \rightarrow 0$ at low temperatures.

While these SMF predictions are correct at $Z = \infty$, they are qualitatively wrong at low temperature in finite connectivity systems. We now turn to a more refined mean field discussion, valid for finite $Z \gg 1$, which is the basis for our main results. We use a quantum version of the cavity method [10] which would become exact if the spins were on a Bethe lattice of connectivity Z . In this method, one studies the properties of a spin j in the cavity graph where one of its neighbours has been deleted, assuming that the $K = Z - 1$ remaining neighbours are uncorrelated. The system of spin j and its K neighbours is thus described by the local Hamiltonian

$$H_j^{cav} = -\xi_j \sigma_j^z - \sum_{k=1}^{Z-1} \left(\xi_k \sigma_k^z + B_k \sigma_k^x + \frac{g}{K} \sigma_j^z \sigma_k^z \right) \quad (4)$$

where B_k is the local ‘‘cavity’’ field on spin k due to the rest of the spins (in absence of j). By solving the problem of Z Ising spins in (4), one can compute the induced magnetization of j , $\langle \sigma_j^x \rangle$, which is by definition equal to $B_j / \sqrt{\xi_j^2 + B_j^2}$. We thus get a mapping allowing to compute the new cavity field B_j in terms of the K fields B_k on the neighbouring spins. This mapping induces a self-consistent equation for

the distribution of the B fields [10]. We have made one more approximation which is to study the cavity Hamiltonian (4) with a mean field method [11]. This gives the explicit mapping:

$$B_i = \frac{g}{K} \sum_{k=1}^K \frac{B_k}{\sqrt{B_k^2 + \xi_k^2}} \tanh \beta \sqrt{B_k^2 + \xi_k^2}. \quad (5)$$

In order to understand this mapping, let us imagine that we iterate it R times on a Bethe lattice. For R finite, when the number of spins is large, the corresponding graph is a rooted tree with branching factor $Z - 1$ at each node and depth R . The field B_0 at the root is a function of the K^R fields on the boundary. In order to see whether there is spontaneous ordering, we study the value of B_0 in linear response to infinitesimal fields $B_i = B \ll 1$ on the boundary spins. This is given by

$$B_0/B = \Xi \equiv \sum_P \prod_{n \in P} \left[\frac{g}{K} \frac{\tanh(\beta\xi_n)}{\xi_n} \right]. \quad (6)$$

where the sum is over all paths going from the root to the boundary, and the product $\prod_{n \in P}$ is over all sites along the path P . The response Ξ is nothing but the partition function for a directed polymer (DP) on a tree, where the energy of each site is $e^{-E_n} = (g/K)(\tanh(\beta\xi_n)/\xi_n)$ and the temperature has been set equal to one. The solution of this problem, found in [13], can be expressed in terms of the convex function $f(x) = (1/x) \log \left[K \int_0^1 d\xi (\tanh(\beta\xi)/\xi)^x \right]$, which is minimal at a value $x = x_c$. In the large R limit, there exist two phases for the DP problem:

- ‘‘Self-averaging’’ (SA) phase: If $x_c > 1$, then $(1/R) \log \Xi = f(1) + \log(g/K)$. The ordered phase appears at $g_c = K e^{-f(1)} = g_c^{SMF}$.
- ‘‘Glassy phase’’ (G) phase: If $x_c < 1$, then $(1/R) \log \Xi = f(x_c) + \log(g/K)$. The ordered phase appears at $g_c = K e^{-f(x_c)} > g_c^{SMF}$.

These two regimes of the DP problem are qualitatively very different. The ‘‘SA’’ regime is the high ‘‘temperature’’ phase of the polymer, where the measure on paths defined in (6) is more or less evenly distributed among all paths. The low temperature ‘‘G’’ regime is a glass phase where the measure condensates onto a small number of paths. An order parameter which distinguishes between these phases is the participation ratio $Y = \sum_P w_P^2$, where w_P is the relative weight of path P in the measure (6). In the replica formalism the SA phase is replica symmetric, Ξ is self-averaging, and $Y = 0$; the G phase is a one-step replica-symmetry-breaking (RSB) glass phase, the value of Y is finite and non self-averaging

(it depends on the explicit realization of the ξ 's even in the thermodynamic limit), and its average is given by $1 - x_c$ [14]. This glass transition, and the nature of the G phase, are identical to the ones found in the random energy model [15, 16].

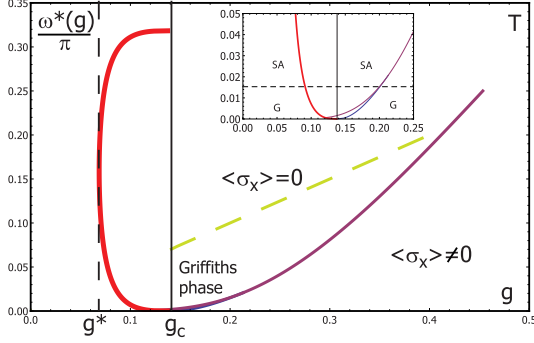


FIG. 1: Phase diagram of the spin system for $K = 2$. The right pane shows the critical line $T_c(g)$ while the left one shows the critical energy that separates the states with zero width from those with a finite width. The inset shows the low temperature/energy region.

Using these DP results one gets the phase diagram of the spin systems shown in the right pane of Fig.1. At any temperature, there is a non-zero critical value of the coupling, $g_c(T)$, separating an “ordered”, superconducting phase with spontaneous x magnetization at $g > g_c(T)$ from a normal “disordered”, insulating phase with zero magnetization at $g < g_c(T)$. Within each phase, there are two regimes of temperature, “Self-averaging” and “Glassy”. As is clear from our susceptibility analysis, the glass transition of the DP affects the propagation of a static perturbation in the spin system. In the disordered SA phase, the total effect of the perturbation decreases when R increases, and propagates evenly: the average value of the susceptibility coincides with its typical value. In the disordered G phase, the total perturbation also decays, but it condenses on a finite number of paths. Consequently, the susceptibility is non-self averaging, similarly to what is found in one dimension[17]. Rare paths are important in the whole “Griffiths” phase where the susceptibility distribution has a power law tail. When $g_c^{SMF} < g < g_c$ the typical susceptibility is finite but the average susceptibility diverges. In the ordered phase, the perturbation propagates to infinity, again with very different patterns in the “SA” and “G” phases. The SMF gets the correct result of the SA regimes, but completely misses the low-temperature physics of condensed correlation paths.

The RSB transition also strongly affects the scaling of the field in the ordered phase, for $g = g_c(1+\epsilon)$. One can study the distribution of fields $P(B)$ in-

duced by the mapping (5). An expansion of its Laplace transform shows that, in the G phase, $P(B)$ decays at large B like $P(B) \simeq C/B^{1+x_c}$. This distribution has a diverging mean, dominated by rare fluctuations. A careful analysis of the self-consistent equation for $P(B)$ then indicates that the typical scale of the field behaves as

$$B_{\text{typ}} \simeq A e^{-g_B/(g-g_c)}. \quad (7)$$

In the disordered phase the average value of the transverse field is zero, but its quantum fluctuations become important. Their main physical effect is the broadening of the local levels that, in the absence of g , correspond to $\sigma_i^z = \pm 1$. At $T = 0$ the level broadening means that local excitations of energy $\omega \simeq 2\xi_i$ decay. If we neglect the phonons, energy conservation requires that some group of spins with the same energy be flipped. In finite systems this is generically impossible. To study whether energy can be transported, and ergodicity can be restored, in infinite systems, we adopt an approach similar to the one developed above. Namely, we consider a Bethe lattice that is very weakly coupled to the environment at its boundary and study the effective level width of a ‘root’ spin at a distance R from the boundary in the $R \rightarrow \infty$ limit. Thus we add to the Hamiltonian (2) the boundary term $H_{\text{env}} = -\sum_j \sigma_j^x x_j(t)$ where $x_j(t)$ are *dynamical* fields generated by the environment, characterized by a response function $G(\omega)$. In the leading order in g/K the relaxation rate of the root spin follows from the Fermi golden rule:

$$\Gamma_0(\omega) = \text{Im } G(\omega) \sum_P \prod_{n \in P} \left[\frac{2g/K}{\omega - 2\xi_k} \right]^2. \quad (8)$$

This perturbation-theory-based equation is only valid when all fractions inside the product remain small[18], and the relaxation rate of each spin is very small. Thus it is self-consistent if $\Gamma_0 \rightarrow 0$ when $R \rightarrow \infty$. The typical value of $\Gamma_0(\omega)$ is controlled by

$$f_\Gamma(\omega) = \frac{1}{R} \ln \left\{ \sum_P \prod_{n \in P} \left[\frac{2}{\omega - 2\xi_k} \right]^2 \right\},$$

it decreases away from the boundary if $f_\Gamma(\omega) + 2 \ln(g/K) < 0$. f_Γ can be computed again by analyzing a DP problem, which turns out to be always in its G phase. At $\omega = 0$, one finds that $f_\Gamma(0) + 2 \ln(g/K) \leq 0$ in the whole insulating regime, and the equality is reached at the critical point $g = g_c$. One also finds that the region of small $\omega - 2\xi$ gives negligible contribution to $f_\Gamma(\omega)$, which allows one to work with the unregularized expression (8). At non-zero frequencies $f_\Gamma(\omega)$ decreases, it is minimal at $\omega = 1/2$ (which corresponds to the center of

the band in our notations). At $g < g^* = Ke^{\text{fr}(1/2)/2}$ the relaxation rate is zero for all states, this is the superinsulator regime of [3]. In the intermediate regime $g_c(0) > g > g^*$ the states in the middle of the band have finite width, they are separated from the zero-width states by a critical energy $\omega^*(g)$ similar to the mobility edge of the non-interacting problem.

In order to understand the low temperature properties of the system, it is important to know the dependence of $\Gamma(\omega)$ at $\omega > \omega^*(g)$. In this regime (8) must be modified. Using the mapping to fermions as in [18], one finds the iterative equation for the width of the levels on the Bethe lattice:

$$\Gamma_i(\omega) = (2g/K)^2 \sum_{k(i)} \frac{\Gamma_k(\omega)}{(\omega - 2\xi_k)^2 + \Gamma_k(\omega)^2} \quad (9)$$

This equation is similar to the equation (5) for the fields in the ordered phase and can be analyzed with the same method, giving the fast level-width dependence slightly above $\omega^*(g)$:

$$\Gamma_{\text{typ}}(\omega) \simeq \Gamma^* e^{-\omega_0(g)/(\omega - \omega^*(g))} . \quad (10)$$

This behaviour has important consequences for the low temperature properties of the relaxation, as we now discuss. A non-zero but low temperature affects the relaxation rate in several ways. First, it changes the occupation numbers of the excited and ground states, this affects the perturbative equation (8) and thus shifts the position of the $\omega^*(g)$ line, this effect is however small at $T \ll 1$. More importantly, a non-zero temperature creates some mobile excitations with frequencies above $\omega^*(g)$. These excitations provide a mechanism for a small level broadening of the very low frequency levels: They see a mobile excitation with energy E with an Arrhenius rate, giving a width $\exp(-\omega_0/(E - \omega^*) - E/T)$. The dominant contribution comes from energies $E = \omega^*(g) + \sqrt{\omega_0 T}$, and results in the temperature dependence $\Gamma(g) \sim \exp(-2\sqrt{\omega_0/T} - \omega^*(g, T)/T)$ that shows a crossover between a square root and activated or even faster behavior as one goes away from the critical point.

In conclusion, we have outlined a solution of the strongly disordered spin model on the Bethe lattice which shows a series of two zero-temperature transitions between a phase with no relaxation, a phase with a slow relaxation and an ordered phase. It also shows that the low temperature phases are always very strongly non-uniform: both the order parameter formation and the spin relaxation are controlled by rare interaction paths containing a very small number of spins. When applied to the superconductor-insulator transition our results imply the existence of both weak and strong insulators. At the critical point the relaxation rate varies

as $\exp(1/\sqrt{T})$ but crosses over to activated at lower g and low T , in the strong insulator the relaxation is completely suppressed. Of course, some of the physical effects neglected in our model would lead to a very slow relaxation even in the strong insulator.

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